IE 598 Inference in Graphical Models

Homework 0

Due-0/0/0000

For the students taking this course, it is expected that you have the skills and backgrounds to solve the problems in this homework (it is encouraged that you talk to your peers to figure the solutions out). If you have trouble following any of these problems, you need to talk to the instructor.

Basic probability.

1. For three discrete random variables x, y, and z with joint probability distribution $p(x,y,z)$, we say x is **conditionally independent** of y given z if and only if $p(x,y|z) = p(x|z)p(y|z)$, where $p(x|z) = \frac{p(x,z)}{p(z)} = \frac{\sum_{y'} p(x,y',z)}{\sum_{x',y'} p(x',y',z)}$ is the conditional probability distribution of x given z, and $p(x,y|z)$ and $p(y|z)$ are defined similarly. We use x–z–y to denote that x and y are independent conditioned on z. We use x–()–y to denote that x is independent of y. Prove each of the following properties.
   (a) x–()–(y,z) implies x–z–y
   (b) x–z–(y,w) implies x–z–y
   (c) x–z–(y,w) and y–z–w implies (x,w)–z–y
2. There are two coins, one is a fair coin and the other is biased. The outcome of tossing a fair coin is a head (H) with probability half. The outcome of tossing a biased coin is a head (H) with probability 3/4. We are given a coin at random (equal probability of getting a fair or biased coin), and want to test whether the coin is biased or not based on n coin tosses. We toss the coin 5 times independently and get (H,H,T,T,H). What is the probability that the coin is a biased coin? Then, what is your **maximum a posteriori (MAP) estimate**? Does it depend on the order of the outcome?

Basic linear algebra.

1. An $n \times n$ dimensional symmetrix matrix A is **positive definite** if $x^T Ax > 0$ for any vector $x \in \mathbb{R}^n$, where x^T is the transpose of the column vector x. It is **positive semidefinite** if $x^T Ax \geq 0$ for any x. Prove that if A has eigen values which are all positive, then A is positive definite.
   [Hint: A symmetric matrix A can be factorized by eigen decomposition as $A = Q\Lambda Q^T$. Q is a unitary matrix such that $QQ^T = Q^TQ = I$, where I is the n-dimensional identity matrix, and Λ is a diagonal matrix with the eigen values λ_i of matrix A in the diagonals. Then, we are left ot show that if Λ is a diagonal matrix with strictly positive entries, then $y^T\Lambda y > 0$, where we changes variables by setting $y = Q^Tx$.].
2. Find a vector y^* in terms of Q_{ij}'s, h_i's and x that maximizes a **quadratic function**

$$f(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

   [Note: the maximizer $y^*(x)$ is a linear function of x.]