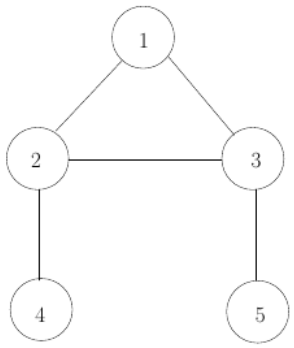
IE 598 Inference in Graphical Models

# Homework 3

Covers lecture slides

4. Elimination algorithm
5. Max-product algorithm
6. Density evolution
7. Gaussian graphical models

**Problem 3.1** Consider the following graphical model.



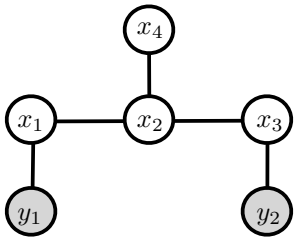

($a$) Draw a factor graph representing the graphical model and specify the factor graph message-passing equations. For this particular example, explain why the factor graph message-passing equations can be used to compute the marginals, but the sum-product equations for pairwise MRF cannot be used.

($b$) Define a new random variable $x_6 = \{x_1, x_2, x_3\}$, i.e., we group variables $x_1$, $x_2$, and $x_3$ into one variable. Draw an undirected graph which captures the relationship between $x_4$, $x_5$, and $x_6$. Explain why you can apply the sum-product algorithm to your new graph to compute the marginals. Compare the belief propagation equations for the new graph with the factor graph message-passing equations you obtained in part ($a$).

($c$) If we take the approach from part ($b$) to the extreme, we can simply define a random variable $x_7 = \{x_1, x_2, x_3, x_4, x_5\}$, i.e., define a new random variable which groups all five original random variables together. Explain what running the sum-product algorithm on the corresponding one vertex graph means. Assuming that we only care about the marginals for $x_1, x_2, \ldots, x_5$, can you think of a reason why we would prefer the method in part ($b$) to the method in this part, i.e., why it might be preferable to group a smaller number of variables together?

**Problem 3.2** Let $x \sim \mathcal{N}^{-1}(h_x, J_x)$, and $y = Cx + v$, where $v \sim \mathcal{N}(0, R)$.

1. Find the potential vector $h_{y|x}$ and the information matrix $J_{y|x}$ of $p(y|x)$.
2. Find the potential vector $h_{x,y}$ and the information matrix $J_{x,y}$ of $p(x, y)$.
3. Find the potential vector $h_{x|y}$ and the information matrix $J_{x|y}$ of $p(x|y)$.
4. Consider the following Gaussian graphical model.




   Let $y_1 = x_1 + v_1$, $y_2 = x_3 + v_2$, and $R = I$ is the identity matrix. Find $C$. Represent messages $h_{x_3 \to x_2}$ and $J_{x_3 \to x_2}$ in terms of $y_2$ and the elements of $h_x$ and $J_x$. [$y_1$ and $y_2$ are measurements, which should be treated as given and deterministically known.]

5. Now assume that we have an additional measurement $y_3 = x_3 + v_3$, where $v_3$ is a zero-mean Gaussian variable with variance 1 and is independent from all other variables. Find the new $C$. Represent messages $h_{x_3 \to x_2}$ and $J_{x_3 \to x_2}$ in terms of $y_2$, $y_3$ and the elements of $h_x$ and $J_x$. [again $y_2$ should be considered as a measurement which is given, and deterministically known.]
6. The BP message from $x_3$ to $x_2$ define a Gaussian distribution with mean $m_{x_3 \to x_2} = J^{-1}_{x_3 \to x_2} h_{x_3 \to x_2}$ and variance $\sigma_{x_3 \to x_2} = J^{-1}_{x_3 \to x_2}$. Comment on the difference in the mean and the variance of this message when computed using a single observation $y_2$ versus when computed using multiple observations $(y_2, y_3)$. Can you guess the mean and variance of the BP message when the number of observations grows to infinity?

**Problem 3.3** In this exercise, you will construct an undirected graphical model for the problem of segmenting foreground and background in an image, and use loopy belief propagation to solve it. Load the image `flower.bmp` into MATLAB using `imread`. (The command `imshow` may also come in handy.) Partial labeling of the foreground and background pixels are given in the mask images `foreground.bmp` and `background.bmp`, respectively. In each mask, the white pixels indicate positions of representative samples of foreground or background pixels in the image. Let $y = \{y_i\}$ be an observed color image, so each $y_i$ is a 3-vector (of RGB values between 0 and 1) representing the pixel indexed by $i$. Let $x = \{x_i\}$, where $x_i \in \{0, 1\}$ 2 is a foreground(1)/background(0) labeling of the image at pixel $i$. Let us say the probabilistic model for $x$ and $y$ given by their joint distribution can be factored in the form

$$\mu(x, y) \quad = \quad \frac{1}{Z} \prod_i \phi(x_i, y_i) \prod_{(j,k) \in E} \psi(x_j, x_k) \tag{1}$$

where $E$ is the set of all pairs of adjacent pixels in the same row or column as in 2-dimensional grid. Suppose that we choose

$$\psi(x_j, x_k) = \begin{cases} 0.9 & \text{if } x_j = x_k \\ 0.1 & \text{if } x_j \neq x_k \end{cases}$$

This encourages neighboring pixels to have the same label–a reasonable assumption. Suppose further that we use a simple model for the conditional distribution $\phi(x_i, y_i) = \mathbb{P}_{Y_i|X_i}(y_i|x_i)$:

$$\mathbb{P}(y_i|x_i = \alpha) \quad \propto \quad \frac{1}{(2\pi)^{3/2}\sqrt{\det\Lambda_\alpha}} \exp\Big\{ -\frac{1}{2}(y_i - \mu_\alpha)^T \Lambda_\alpha^{-1}(y_i - \mu_\alpha)\Big\} + \epsilon$$

for $y_i \in [0,1]^3$. That is, the distribution of color pixel values over the same type of image region is a modified Gaussian distribution, where $\epsilon$ accounts for outliers. Set $\epsilon = 0.01$ in this problem.

(*a*) Sketch an undirected graphical model that represents $\mu(x, y)$.

(*b*) Compute $\mu_\alpha \in \mathbb{R}^3$ and $\Lambda_\alpha \in \mathbb{R}^{3\times 3}$ for each $\alpha \in \{0, 1\}$ from the labeled masks by finding the sample mean and covariance of the RGB values of those pixels for which the label $x_i = \alpha$ is known from `foreground.bmp` and `background.bmp`. The sample mean of samples $\{y_1, \ldots, y_N\}$ is $\bar{y} = \frac{1}{N}\sum_{i=1}^N y_i$ and the sample covariance is $C_y = \frac{1}{N-1}\sum_{i=1}^N (y_i - \bar{y})(y_i - \bar{y})^T$.

(*c*) We want to run the sum-product algorithm on the graph iteratively to find (approximately) the marginal distribution $\mu(x_i|y)$ at every $i$. For a joint distribution of the form (1) with pairwise compatibility functions and singleton compatibility functions, the local message update rule for passing the message $\nu_{j\to k}(x_j)$ from $x_j$ to $x_k$, is represented in terms of the messages from the other neighbors of $x_j$, the potential functions.

$$\nu_{j\to k}(x_j) \propto \phi(x_j, y_j) \prod_{u\in\partial j\setminus k} \sum_{x_u} \psi(x_j, x_u)\nu_{u\to j}(x_u)$$

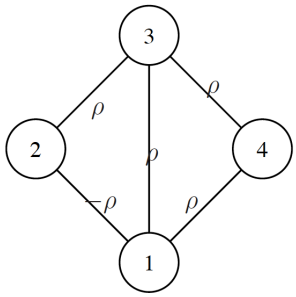
Then the final belief on $x_j$ is computed as

$$\nu_j(x_j) \propto \phi(x_j, y_j) \prod_{u\in\partial j} \sum_{x_u} \psi(x_j, x_u)\nu_{u\to j}(x_u)$$

Implement the sum-product algorithm for this problem. There are four directional messages: down, up, left, and right, coming into and out of each $x_i$ (except at the boundaries). Use a parallel update schedule, so all messages at all $x_i$ are updated at once. Run for 30 iterations (or you can state and use some other reasonable termination criterion). Since we are working with binary random variables, perhaps it is easier to pass messages in log-likelihood. Feel free to use `gridbpsol.m` from the website for running the BP algorithm.
After the marginal distributions at the pixels are estimated, visualize their expectation. Where are the beliefs "weak"?
Visualize the expectation after 1, 2, 3, and 4 iterations. Qualitatively, discuss where the loopy belief propagation converge first and last.
Run BP with a different value of $\epsilon = 0$ and comment on the result.
Run BP with a different pairwise potential and comment on the result.

$$\psi(x_j, x_k) = \begin{cases} 0.6 & \text{if } x_j = x_k \\ 0.4 & \text{if } x_j \neq x_k \end{cases}$$

**Problem 3.4** Consider the Gaussian graphical model depicted below. More precisely, if we let $x$ denote the 4-dimensional vector of variables at the 4 nodes (ordered according to the node numbering given), then $x \sim \mathcal{N}^{-1}(h, J)$, where J has diagonal values all equal to 1 and non-zero off-diagonal entries as indicated in the figure (e.g., $J_{12} = -\rho$).



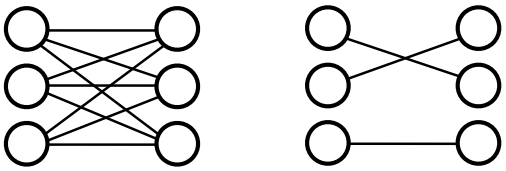

(a) Confirm (e.g., by checking Sylvesters criterion to see if the determinants of all principal minors are positive) that $J$ is a valid information matrix–i.e., it is positive definite–if $\rho = .39$ or $\rho = .4$. Compute the variances for each of the components (i.e., the diagonal elements of $\Lambda = J^{-1}$)–you can use Matlab to do this if youd like.

(b) We now want to examine Loopy BP for this model, focusing on the recursions for the information matrix parameters. Write out these recursions in detail for this model. Implement these recursions and try for $\rho = .39$ and $\rho = .4$. Describe the behavior that you observe.

(c) Construct the computation tree for this model. Note that the effective "$J$" – parameters for this model are copies of the corresponding ones for the original model (so that every time the edge $(1, 2)$ appears in the computation tree, the corresponding $J$-component is $-\rho$). Use Matlab to check the positive-definiteness of these implied models on computation trees for different depths and for our two different values of $\rho$. What do you observe that would explain the result in part (b)?

**Problem 3.5** In this problem, we apply inference techniques in graphical models to find *Maximum Weight Matching (MWM)* in a complete bipartite graph. This is one of a few problems where belief propagation converges and is correct on a general graph with loops. Other such examples include Gaussian graphical models studied in class.

Consider an undirected weighted complete bipartite graph $G(X, Y, E)$ where $X$ is a set of $n$ nodes and $Y$ is another set of $n$ nodes: $|X| = |Y| = n$. In a bipartite complete graph all the nodes in $X$ are connected to all the nodes in $Y$ and vice versa, as shown below. Further, each edge in this graph is associated with a real



a complete bipartite graph  a perfect matching

valued weight $w_{ij} \in \mathbb{R}$. A *matching* in a graph is a subset of edges such that no edges in this matching share a node. A matching is a *perfect matching* if it matches all the nodes in the graph. Let $\pi = (\pi(1), \ldots, \pi(n))$ be a permutation of $n$ nodes. In a bipartite graph, a permutation $\pi$ defines a perfect matching $\{(i, \pi(i)\}_{i \in \{1,\ldots,n\}}$. From now on, we use a permutation to represent a matching. A *weight* of a (perfect) matching is defined as $W_\pi = \sum_{i=1}^{n} w_{i,\pi(i)}$. The problem of *maximum weight matching* is to find a matching such that

$$\pi^* \quad = \quad \arg\max_{\pi} W_\pi \ .$$

We want to solve this maximization by introducing a graphical model with probabilty proportional to the weight of a matching:

$$\mu(\pi) \quad = \quad \frac{1}{Z}\, e^{CW_\pi}\, \mathbb{I}(\pi \text{ is a perfect matching}) \,,$$

for some constant $C$.

($a$) The set of matchings can be encoded by a pair of vectors $x \in \{1,\ldots,n\}^n$ and $y \in \{1,\ldots,n\}^n$, where each node takes an integer value from 1 to $n$. With these, we can represent the joint distribution as a pari-wise graphical model:

$$\mu(x,y) = \frac{1}{Z} \prod_{(i,j)\in\{1,\ldots,n\}^2} \psi_{ij}(x_i,y_j) \prod_{i=1}^{n} e^{w_{i,x_i}} \prod_{i=1}^{n} e^{w_{y_i,i}} \,,$$

where $\psi_{ij}(x_i,y_j) = \begin{cases} 0 & x_i = j \text{ and } y_j \neq i \,, \\ 0 & x_i \neq j \text{ and } y_j = i \,, \\ 1 & \text{otherwise} \,. \end{cases}$ Show that for the pairwise graphical model defined above, the joint distribution $\mu(x,y)$ is non-zero if and only if $\pi_x = \{(1,x_1),\ldots,(n,x_n)\}$ and $\pi_y = \{(y_1,1),\ldots,(y_n,n)\}$ both are matchings and $\pi_x = \pi_y$. Further, show that when non-zero, the probability is equal to $\frac{1}{Z} e^{2W_{\pi_x}}$.

($b$) Let

$$(x^*,y^*) \quad = \quad \arg\max_{x,y}\, \mu(x,y) \,.$$

Show that $\pi_{x^*} = \pi_{x^*}$ is the maximum weight matching on the given graph $G$ with weights $\{w_{ij}\}$.

($c$) Let us denote by $l_i$ the $i$-th 'left' node in $X$ corresponding to the random variable $x_i$, and by $r_j$ the $j$-th 'right' node in $Y$ corresponding to the random variable $y_j$. We are going to derive *max-product* update rules for this problem. Let $\nu_{l_i\to r_j}(x_i)^{(t)}$ denote the message from a left node $l_i$ to a right node $r_j$ at $t$-th iteration, which is a vector of size $n$. Similarly, we let $\nu_{r_j\to l_i}(y_j)^{(t)}$ denote the message from a right node $r_j$ to a leftt node $l_i$. We initialize all the messages such that

$$\begin{aligned} \nu^{(0)}_{l_i\to r_j}(x_i) \quad &= \quad e^{w_{i,x_i}} \,, \\ \nu^{(0)}_{r_j\to l_i}(y_j) \quad &= \quad e^{w_{y_j,j}} \,. \end{aligned}$$

Write the message update rule for the message $\nu^{(t+1)}_{l_i\to r_j}(x_i)$ and $\nu^{(t+1)}_{r_j\to l_i}(y_j)$ as functions of messages from previous iterations.

**Problem 3.6** [Optional] As mentioned in class, Gaussian BP allows to compute the minimum of a quadratic function

$$\widehat{x} = \arg\min_{x\in\mathbb{R}^n} \left\{ \frac{1}{2}\langle x, Qx\rangle + \langle b, x\rangle \right\}. \tag{2}$$

for $Q \in \mathbb{R}^{n\times n}$ positive definite, where $\langle a,b\rangle = a^T b$ indicates the standard inner product of two vectors. In this homework we will consider a case in which $Q$ is not positive definite, but is symmetric and has full rank. in this case we can still define

$$\widehat{x} = -Q^{-1} b \,. \tag{3}$$

which is a stationary point (a saddle point) of the above quadratic function. The BP update equations are exactly the same as for the minimization problem with a positive definite $Q$. We claim that, when BP converges, it still computes the correct solution $\widehat{x}$.

We consider a specific model. An unknown signal $s_0 \in \mathbb{R}^n$ is observed in Gaussian noise

$$y = As_0 + w_0 \,. \tag{4}$$

Here $y \in \mathbb{R}^m$ is a vector of observations, $A \in \mathbb{R}^{m \times n}$ is a measurement matrix, and $w_0 \in \mathbb{R}^m$ is a vector of Gaussian noise, with i.i.d. entries $w_{0,i} \sim \mathcal{N}(0, \sigma^2)$. We are given $y$ and $A$, and would like to reconstruct the unknown vector $s_0$, and hence $w_0$.

A popular method consists in solving the following quadratic programming problem (known as *ridge regression*):

$$\widehat{s} = \arg\min_{s \in \mathbb{R}^n} \left\{ \frac{1}{2} \big\| y - As \big\|_2^2 + \frac{1}{2} \lambda \|s\|_2^2 \right\} . \tag{5}$$

We will do something equivalent. For $x \in \mathbb{R}^{m+n}$, $x = (z, s)$, $z \in \mathbb{R}^m$, $s \in \mathbb{R}^n$, we define a cost function

$$\mathcal{C}_{A,y}(x = (z,s)) \quad = \quad -\frac{1}{2}\|z\|_2^2 + \frac{1}{2}\lambda \|s\|_2^2 + \langle z, y - As \rangle \,. \tag{6}$$

We will look for the stationary point of $\mathcal{C}_{A,y}$.

($a$) Show that the cost function $\mathcal{C}_{A,y}(x)$ can be written in the form

$$\mathcal{C}_{A,y}(x) = \frac{1}{2}\langle x, Qx \rangle + \langle b, x \rangle \,. \tag{7}$$

Write explicitly the form of the matrix $Q \in \mathbb{R}^{(m+n)\times(m+n)}$ and the vector $b \in \mathbb{R}^{m+n}$.

($b$) Let $\widehat{x} = (\widehat{z}, \widehat{s})$ be the stationary point of $\mathcal{C}_{A,y}(z, s)$. Assuming it is unique, show that $\widehat{s}$ does coincide with the ridge estimator (5).

($c$) Write the update rule for the BP algorithm (equivalent to the sum-product algorithm) to compute the stationary point $\widehat{x} = (\widehat{z}, \widehat{s})$ of $\mathcal{C}_{A,y}(x)$. [hint: use the same ideas from the Gaussian belief propagation for positive definite $Q$.]

($d$) Prove the above claim that, if BP converges, then it computes $\widehat{x}$, cf. Eq. (3) even if $Q$ is not positive definite.

**Problem 3.7** [Density Evolution] In this problem we consider using Low-Density Parity Check (LDPC) codes to encode bits to be sent over a noisy channel.

**Encoding.** LDPC codes are defined by a factor graph model over a bipartite graph $G(V, F, E)$, where $V$ is the set of variable nodes, each representing the bit to be transmitted, and $F$ is a set of factor nodes describing the code and $E$ is a west of edges between a bit-node and a factor node. The total number of variable nodes in the graph define the length of the code (also known as the block length), which we denote by $n \triangleq |V|$.

We consider binary variables $x_i \in \{-1, +1\}$ for $i \in V$, and all codewords that are transmitted satisfy

$$\prod_{i \in \partial a} x_i \quad = \quad +1 \,,$$

which means that there are even number of $-1$'s in the neighborhood of any factor node.

**Channel.** We consider a Binary Symmetric Channel, known as BSC($\varepsilon$), where one bit is transmitted over the channel at each discrete time step, and each transmitted bit is independently flipped with probability $\varepsilon$. Precisely, let $x_i \in \{+1, -1\}$ be a transmitted bit and $y_i \in \{+1, -1\}$ be the received bit (at time $i$), then

$$\begin{aligned}
\mathbb{P}(y_i = +1 | x_i = +1) &= 1 - \varepsilon \,, \\
\mathbb{P}(y_i = -1 | x_i = +1) &= \varepsilon \,, \\
\mathbb{P}(y_i = -1 | x_i = -1) &= 1 - \varepsilon \,, \\
\mathbb{P}(y_i = +1 | x_i = -1) &= \varepsilon \,.
\end{aligned}$$

The conditional probability distribution over $x_1^n = [x_1, \ldots, x_n]$ given the observed received bits $y_1^n = [y_1, \ldots, y_n]$ is

$$\mu(x_1^n \,|\, y_1^n) \;=\; \frac{1}{Z} \prod_{i \in V} \psi_i(x_i, y_i) \prod_{a \in F} \mathbb{I}(\otimes x_{\partial a} = +1) \,,$$

where $\psi_i(x_i, y_i) = \mathbb{P}(y_i | x_i)$ and $\otimes$ indicates product of binary numbers such that if $x_{\partial a} = \{x_1, x_2, x_3\}$ then $\otimes x_{\partial a} = x_1 \times x_2 \times x_3$ (to be precise we need to take $\psi_i(x_i | y_i) = \mathbb{P}(x_i | y_i)$, but this gives the exactly same conditional distribution as above since any normalization with respect to $y_i$'s are absorbed in the partition function $Z$). This is naturally a graphical model on a factor graph $G(V, F, E)$ defined by the LDPC code.

(*a*) Write down the belief propagation updates (also known as the (parallel) sum-product algorithm) for this factor graph model for the messages $\{\nu_{i \to a}^{(t)}(\cdot)\}_{(i,a) \in E}$ and $\{\tilde{\nu}_{a \to i}^{(t)}(\cdot)\}_{(i,a) \in E}$ .

(*b*) What is the computational complexity (how many operations are required in terms of the degrees of the variable and factor nodes) for updating one message $\nu_{i \to a}(\cdot)$ and one message $\tilde{\nu}_{a \to i}(\cdot)$ respectively? Explain how one can improve the computational complexity, to compute the message $\tilde{\nu}_{a \to i}^{(t)}(\cdot)$ exactly in runtime $O(d_a)$, where $d_a$ is the degree of the factor node $a$.

(*c*) Now, we consider a different message passing algorithm introduced by Robert Gallager in 1963. The following update rule is a message passing algorithm known as the **Gallager A algorithm**. Similar to the belief propagation for BEC channels we studied in class, this algorithm also sends discrete messages (as opposed to real-valued messages in part (*a*)). Both $\nu_{i \to a}^{(t)}$'s and $\tilde{\nu}_{a \to i}^{(t)}$'s are binary, i.e. in $\{+1, -1\}$.

$$\begin{aligned}
\nu_{i \to a}^{(t+1)} &= \begin{cases} +1 & \text{if } \tilde{\nu}_{b \to i}^{(t)} = +1 \text{ for all } b \in \partial i \setminus a \,, \\ -1 & \text{if } \tilde{\nu}_{b \to i}^{(t)} = -1 \text{ for all } b \in \partial i \setminus a \;\; , \\ y_i & \text{otherwise} \,, \end{cases} \\
\tilde{\nu}_{a \to i}^{(t)} &= \prod_{j \in \partial a \setminus i} \nu_{j \to a}^{(t)} \,.
\end{aligned}$$

The interpretation of this update rule is that $\nu_{i \to a}$ messages trust the received bit $y_i$ unless all of the incoming messages disagree with $y_i$, and $\tilde{\nu}_{a \to i}$ messages make sure that the consistency with respect to $\mathbb{I}(\otimes x_{\partial a})$ is satisfied. In this algorithm, the messages take values in $\{+1, -1\}$ and are the estimated values of $x_i$'s, as opposed to the distribution over those values as in belief propagation.

We assume that random $(\ell, r)$-regular bipartite graph is used to generate the LDPC code. In the resulting random graph, all variable nodes have degree $\ell$ and all factor nodes have degree $r$. Among all such graphs, a random graph is selected uniformly at random.

Define $W^{(t)}$ to be the (empirical) distribution of the messages $\{\nu_{i \to a}^{(t)}\}_{(i,a) \in E}$ and $Z^{(t)}$ to be the (empirical) distribution of the messages $\{\tilde{\nu}_{a \to i}^{(t)}\}_{(i,a) \in E}$. We assume the messages are initialized in such way that $\nu_{i \to a}^{(0)} = y_i$ for all $i \in V$. We also assume, without loss of generality, that all $+1$ messages were

sent, i.e. $x_i = +1$ for all $i$. Then, let $w^{(t)} = \mathbb{P}(W^{(t)} = -1)$ be the probability that a message $\nu_{i\to a}^{(t)}$ is $-1$ for a randomly chosen edge $(i, a)$, and let $z^{(t)} = \mathbb{P}(Z^{(t)} = -1)$ be the probability that a message $\tilde{\nu}_{a\to i}^{(t)}$ is $-1$ for a randomly chosen edge $(i, a)$.

*Write the* **density evolution** *equations for $w^{(t)}$ and $z^{(t)}$, describing how the random distribution of the messages $w^{(t)}$ and $z^{(t)}$ evolve.* [We are looking for a clean answer. Specifically, the number of operations required to compute $z^{(t)}$ from $w^{(t)}$ should be $O(1)$. The same technique that reduced computation in part $(b)$ should be helpful.]

$(d)$ Write the density evolution equation for a single scalar variable $w^{(t)}$, by substituting $z^{(t)}$. This gives a fixed point equation in the form of $w^{(t)} = F(w^{(t-1)})$ for some $F$. Plot (using MATLAB to your favorite numerical analysis tool) the function $y = F(x)$ and the identify function $y = x$, for $\ell = 3$ and $r = 4$, and for two values of $\varepsilon = 0.05$ and $\varepsilon = 0.1$ Explain the figure in terms of the error probability of the (3,4)-code on those two BSC($\varepsilon$)'s.