8. Cake cutting

- proportionality and envy-freeness
- price of fairness
- approximate envy-freeness

Cake cutting

model

- a cake is the interval $[0, 1]$
- a peice of cake is $X \subseteq [0, 1]$
- each of n agents has valuation V_i over X such that
  - ★ normalized: $V_i([0, 1]) = 1$
  - ★ additive: $V_i(X \cup Y) = V_i(X) + V_i(Y)$ for all $X \cap Y = \emptyset$
  - ★ divisible: for all $\lambda \in [0, 1]$, exists a piece $X' \subseteq X$ such that $V_i(X') = \lambda V_i(X)$

problem: find a partition $A = \{A_1, \ldots, A_n\}$ that is *fair*

notion of fairness

proportionality: for all $i \in [n]$

$$V_i(A_i) \geq \frac{1}{n}$$

envy-freeness(EF): for all $i, j \in [n]$

$$V_i(A_i) \geq V_i(A_j)$$

in general EF implies proportionality but not the other way (EF is equivalent as proportionality when $n = 2$)

protocol satisfying *envy-freeness* for $n = 2$

cut-and-choose algorithm for $n = 2$

1. agent 1 divides the cake into two equally evaluated pieces such that $V_1(X) = V_1(X^c) = 1/2$
2. agent 2 chooses the preferred piece

claim. cut-and-choose algorithm is EF, and hence proportional

Robertson-Webb model for complexity

two types of queries are allowed

- ⋆ $Eval(i, x, y) = V_i([x, y])$
- ⋆ $Cut(i, x, \alpha) = y$ such that $V_i([x, y]) = \alpha$

e.g. cut-and-choose algorithm requires 2 queries

two protocols satisfying *proportionality* for general $n \geq 3$

Dubins-Spanier protocol

- ⋆ for $i = 1 : n$
- ⋆ remaining agents mark $1/n$ point from the left of the remaining cake
- ⋆ the leftmost agent gets the piece up to his marks
- ⋆ the agent who got the piece is removed and the piece is also removed
- ⋆ end for.

claim. DS protocol satisfies proportionality

complexity: DS protocol requires $O(n^2)$ queries

Even-Paz protocol

- ★ given $[x_1, x_2] = [0, 1]$
- ★ if $n > 1$
- ★ each agent marks z_i such that $V_i([x_1, z_i]) = \frac{1}{2} V_i([x_1, x_2])$
- ★ let x^* be the $n/2$-th mark from the left
- ★ else if $n = 1$
- ★ the agent gets the piece $[x_1, x_2]$
- ★ repeat with left $n/2$ agents on $[x_1, x^*]$ and right agents on $[x^*, x_2]$

claim. EP protocol satisfies proportionality

complexity: DS protocol requires $O(n \log n)$ queries

theorem.[Edmonds,Pruhs '06] Any proportional protocol requires $\Omega(n \log n)$ queries

protocol satisfying *envy-freeness* for $n = 3$

Selfridge-Conway protocol for three agents $\{A, B, C\}$

- stage 0 (divide the cake into two parts)
  - A divides the cake into three equal pieces according to V_A
  - B trims the largest piece according to V_B such that the two largest pieces have the same value for B
  - let the trimmed part be Cake 2 and the reamining part be Cake 1
- stage 1 (allocate the cake 1)
  - Cake 1 is already divided into three pieces X_1, X_2, X_3 such that $1/3 = V_A(X_1) = V_A(X_2) \geq V_A(X_3)$ (we let X_3 be the trimmed piece)
  - we also know that $V_B(X_1) \leq V_B(X_2) = V_B(X_3)$ (we let X_1 be the piece that is smallest for B)
  - C chooses the most preferred piece among X_1, X_2, X_3
  - B chooses the most preferred piece among the rest (but when there is a tie, he chooses X_3)
  - A gets the remaining piece
- stage 2 (allocate the cake 2)
  - let $T \in \{B, C\}$ be the agent who got the trimmed piece X_3
  - let $T' \in \{B, C\}$ be the agent who did not get the trimmed piece X_3
  - agent T chooses the most preferred piece among the three
  - A chooses the most preferred piece among the rest
  - agent T' gets the remaining piece

claim. SC protocol satisfies envy-freeness

proof.

in stage 1 for the division of cake 1

- ⋆ C is envy-free since he chooses first
- ⋆ B is envy-free since he always gets either X_2 or X_3 that have the same value for him
- ⋆ A is envy-free since he always gets either X_1 or X_2 that have the same value for him

in stage 2 after the division of cake 2

- ⋆ T is envy-free since he chooses first (and he was envy free in stage 1)
- ⋆ A is envy-free of T' because he chooses before T'
- ⋆ A is envy-free of T, since $V_A(X_1) = V_A(X_2) = 1/3$ is the value of his piece from stage 1, and it is at least the value of what T got in stage 1 and the whole cake 2, i.e. $V_A(X_3) + V_A(\text{cake } 2) = 1/3$
- ⋆ T' is envy free since all the pieces have the same value for him (and he was envy free in stage 1)

in general for $n \geq 4$, the best known protocol might require unbounded number of queries [Brams, Taylor, 1995]

The price of fairness

let *Social Welfare* of a allocation A be

$$\sum_{i=1}^{n} V_i(A_i)$$

the *Price of envy-freeness* is the worst-case ratio between the social welfare of the best allocation and the best EF allocation

$$\max_{\{V_1,\ldots,V_n\}} \frac{\max_{A} \sum_{i=1}^{n} V_i(A_i)}{\max_{B \text{ satisfying EF}} \sum_{i=1}^{n} V_i(B_i)}$$

theorem. [Caragiannis et al. '09] the price of envy-freeness is $\Omega(\sqrt{n})$
proof.

- ★ agents $B = \{1, \ldots, \sqrt{n}\}$ desire disjoint intervals of length $1/\sqrt{n}$ uniformly
- ★ the rest B^c desire the whole cake uniformly
- ★ social welfare maximized by giving the cake to B s.t. $\sum_i V_i(A_i) = \sqrt{n}$
- ★ EF allocation must give at least $\frac{n-\sqrt{n}}{n}$ fraction of cake to B^c, resulting $\sum_i V_i(A_i) \leq \frac{n-\sqrt{n}}{n} + \sqrt{n}\frac{\sqrt{n}}{n} \leq 2$

suppose we allocate the cake to n agents and for each we must allocate connected pieces

the dumping paradox: throwing away a part of the cake can increase the social wellfare of the best EF allocation by a factor of $\sqrt{n}$ [Azri et al. 2011]

example with two agents where the social welfare increases from 1 to arbitrarily close to 1.5

- agent 1 desires the whole cake uniformly
- agent 2 desires the middle ϵ interval $[\frac{1-\epsilon}{2}, \frac{1+\epsilon}{2}]$ of the cake uniformly
- only one EF and connected cut at $1/2$, which achieves socail welfare of 1
- if we throw away the piece $[0, \epsilon]$, and allocate the piece $[\epsilon, \frac{1+\epsilon}{2}]$ to agent 2 and $[\frac{1+\epsilon}{2}, 1]$ to agent 1, then this is EF and connected
- further, this allocation achieves social welfare of $1.5 - \epsilon$

Approximate envy-freeness

since there is no known EF protocol with finite complexity, efficient approximate protocols have been proposed such that

$$V_i(A_i) \;\geq\; V_i(A_j) - \varepsilon \;,$$

for all $i, j \in [n]$

suppose there are m indivisible goods, and we partition these goods into n subsets $A = \{A_1, \dots, A_n\}$ to be allocated to n agents with value function $V_i(\cdots)$'s, and let

$$\begin{aligned} e_{ij}(A) &= \max\{0, V_i(A_j) - V_i(A_i)\} \\ e(A) &= \max_{i,j \in [n]} e_{ij}(A) \end{aligned}$$

and define the *maximum marginal utility* as

$$\alpha \;\equiv\; \max_{i, A_i, x} \{ V_i(A_i \cup \{x\}) - V_i(A_i) \}$$

in the case of *additive* value functions (as in the cake cutting setting), the maximum marginal utility simplifies to $\alpha = \max_{i,x} V_i(x)$

theorem. [Lipton et al. 2004] given m indivisible goods and n agents with value functions V_i's, there is a polynomial time algorithm for finding an allocation satisfying

$$e(A) \leq \alpha$$

- we can use the above theorem (and the algorithm that will be explained in the proof of the theorem in the following) to find an ε-approximate EF allocation in polynomial time
  - ★ each agent i makes $\lceil \frac{1}{\varepsilon} \rceil$ marks $x_1^{(i)}, \ldots, x_{\lceil 1/\varepsilon \rceil}^{(i)}$ such that each interval is ε valuable, i.e. $V_i([x_k^{(i)}, x_{k+1}^{(i)}]) = \varepsilon$ for $i = 1, \ldots, \lceil \frac{1}{\varepsilon} \rceil - 1$
  - ★ given $n\lceil 1/\varepsilon \rceil$ marks, one can treat each of the $1 + (n\lceil 1/\varepsilon \rceil)$ intervals as indivisible goods such that the maximum marginal utility is bounded by $\alpha \leq \varepsilon$
  - ★ applying the above theorem with $m = 1 + (n\lceil 1/\varepsilon \rceil)$ goods, we get an ε-EF protocol with polynomial running time

Proof of the Lipton's theorem.
given an allocation A, consider an *envy graph* $G_A(V, E)$ s.t. we have a directed edge (i, j) if i envies j, i.e. $e_{ij} = \max\{0, V_i(A_j) - V_i(A_i)\}$

lemma. For any envy graph G_A with allocation A which might not inlude all the goods, there exists an allocation B of all the goods that are included in A with envy graph G_B such that G_B is has no cycles and the approximation gap does not increase such that $e(B) \leq e(A)$

using the above lemma, we can iteratively allocate the goods while maintaining the weight of the edges at most α

1. Initialize A as allocating all agents emptysets

$$A^{(0)} = \{A_1^{(0)} = \emptyset, \ldots, A_n^{(0)} = \emptyset\}$$

2. For $k = 1, \ldots, m$
   - allocate the k-th good g_k to (one of) the souce node of the graph G_A, where a source is a node that has no incoming edge and call this allocation $B^{(k)}$
   - find a new allocation $A^{(k)}$ of the first k goods by eliminating the cycle using the above lemma
3. end for.

this algorithm maintains $e(A^{(k)}) \leq \alpha$, since

(a) $e(A^{(0)} = 0$;
(b) $e(B^{(k)}) \leq e(A^{(k-1)})$; and
(c) $e(A^{(k)}) \leq e(B^{(k)})$;

where (c) follows from the lemma and
(b) follows from the fact that when we add a good to a source node i, the resulting graph can only add edges that end at node i since node i's allocation only increased and all the other nodes allocations stay the same
further, since i is the source in the original graph, there is no incoming edge, hence

$$e_j i(B^{(k)}) \leq e_{ji}(A^{(k-1)}) + \alpha = \alpha$$

and this finishes the proof of the theorem

proof of the lemma.

key insight: if there is a cycle in G_A, then shift the allocations along the cycle to obtain a new allocation A' such that $e(A) \leq e(A')$

* ⋆ for example, a cycle of two agents i and j implies $V_i(A_i) \leq V_i(A_j)$ and $V_j(A_j) \leq V_j(A_i)$
* ⋆ then by giving A_j to agent i and A_i to j, everyone is better off, in the sense that both i and j are allocated a better piece than before and all the other agents stay the same, hence

$$e(A') \leq e(A)$$

and when we remove a cycle, the number of edges only decrease

we can iteratively remove cycles until all cycles are gone

when valuations are additive (as is the case for cake cutting problem), there is an alternative and simpler way to assign goods while maintaining envy $\leq \varepsilon$

* ⋆ create indiviiisible goods in the same way
* ⋆ agents choose most preferred pieces in a round-robin fashion: $1, 2, \ldots, n, 1, \ldots, n, \ldots$
* ⋆ each good chosen by agent i at k-th round is preferred to the good chosen by agent j in $k+1$-th round
* ⋆ hence, envy is at most the value of a good chosen in the first round, which is at most ε