IE 512 Graphs, Networks, and Algorithms

Midterm

Some problems are more difficult than the others, so read all the problems before you begin to solve them.

Problem 1 There is a postman, who parks his car at a place and wants to walk around delivering mail to every street and return to the car. He wants to minimize the total number of streets that he visits. This can be formulated as a graph problem: given a connected undirected graph $G = (V, E)$, find a closed walk of minimum length that traverses every edge at least once. The length of a walk is defined as the number of edges included in that walk.

(a) Using Eulerian cycle, give a polynomial time algorithm that gives a closed walk of length at most $2|E|$.

Given an undirected weighted graph $G = (V, E)$ with weights w_{ij}'s, we want to find the shortest possible route (closed walk) that visits each node at least once and returns to a start node.

(b) Using Minimum spanning tree, give a polynomial time algorithm that gives a closed walk of length at most $2(\sum_{(i,j)\in \text{MST}} w_{ij})$, where MST is the set of edges in the minimum spanning tree.

Problem 2 (Birkhoff-von Neumann theorem)

A *double stochastic* matrix is a square matrix with non-negative entries whose row sums and column sums are all ones (the row sum of the i-th row is the sum of the entries in that row). A *magic square* is a square matrix with non-negative *integer* entries whose row sums and columns sums are all equal. We call this common value of the row and columns sums, the *weight* of a magic square. Given a magic square of weight d, if we divide all the entries of the magic square by d, we get a doubly stochastic matrix. Conversely, given a doubly stochastic matrix with *rational* entries, the multiplying the entries of this double stochastic matrix by the least common denominator gives a double stochastic matrix with weight equal to that least common denominator.

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

a magic square of weight 3 (left) a doubly stochastic matrix (middle) and a permutation matrix (right)

A magic square of weight one is called a *permutation matrix*, since for any permutation matrix $T \in \mathbb{R}^{m\times m}$, there exists a permutation σ over $[m] = \{1, \ldots, m\}$ such that

$$T_{i,j} = \begin{cases} 1 & \text{if } j = \sigma(i) \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

In other words, there is a one-to-one correlation between a permutation and a permutation matrix.

We want to prove the following Birkhoff-von Neumann theorem, step by step.

Theorem 0.1. *Every doubly stochastic matrix is a convex combination of permutation matrices. Every magic square of weight d is the sum of d (not necessarily distinct) permutation matrices.*

We only consider the magic squares, since the proof for the doubly stochastic matrices follows similarly.

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

an example of Birkhoff-von Neumann theorem

(a) Given an $m \times m$ magic square M of weight d, consider a bipartite graph $G = (A, B, E)$ where $A = \{1, 2, \dots, m\}$ is the set of nodes corresponding to the rows of M, $B = \{1, 2, \dots, m\}$ is the set of nodes corresponding to the columns of M, and $E = \{(i, j) : M_{i,j} > 0\}$. For any magic square M, prove that for subset $X \subseteq A$,

$$|X| \le |N(X)| .$$

(b) Applying the Hall's matching theorem, (a) implies that for any magic square M, there exists a perfect matching in the bipartite graph G derived from M. Use this statement (inductively) to prove Theorem 0.1.

Problem 3 For an undirected graph $G = (V, E)$, let $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n$ be the eigenvalues of the adjacency matrix A, where

$$A_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Let $d_{\text{ave}} = \frac{1}{n} \sum_i d_i$ be the average degree of the graph and $d_{\max}$ be the maximum degree. We know from homework that

$$d_{\text{ave}} \le \lambda_1 \le d_{\max} .$$

(a) Let B be a symmetric matrix obtained from A by removing the i-th row and the i-th column from it, for some integer $i \in [n]$. Let $\mu_1 \ge \dots \ge \mu_{n-1}$ be the eigenvalues of B. Prove that

$$\lambda_1 \ge \mu_1 .$$

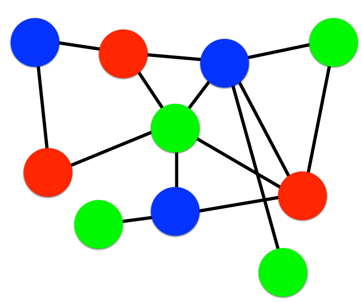

Figure 1: an example of 3-colorable graph

(*b*) An undirected graph $G = (V, E)$ is k-colorable if the nodes can be colored by k colors such that no adjacent nodes have the same color. Precisely, a k-colorable graph has a k way partition $P_1, \ldots, P_k \subseteq V$ such that $P_i \cap P_j = \emptyset$ for all $i, j \in [k]$ and $\cup_{i=1}^{k} P_k = V$ satisfying

$$(i, j) \notin E \text{ if } i \in P_\ell \text{ and } j \in P_\ell \text{ for some } \ell \in [k] \; .$$

Prove that a graph with the maximum degree $d_{\max}$ is $(d_{\max} + 1)$-colorable.

(*c*) Using induction on the size of the graph, prove that an undirected graph with largest eigenvalue of the adjacency matrix λ_1 is $(\lfloor \lambda_1 \rfloor + 1)$-colorable.

Problem 4 Consider a natural random walk on an aperiodic and connected undirected graph $G = (V, E)$ with n nodes. The random walk follows the transition matrix $P = AD^{-1}$ where A is the adjacency matrix and D is a diagonal matrix with degree of the nodes in the diagonals. The stationary distribution of this random walk is $\pi = \frac{1}{\sum_{i=1}^{n} d_i} d$ where $d = [d_1, \ldots, d_n]^T$ is a vector in $\mathbb{R}^n$.

Now, consider the symmetric matrix $M = D^{-1/2} A D^{-1/2}$. It is well known fact in linear algebra that symmetric matrices can be diagonalized as $M = U \Lambda U^T$, where U is an orthogonal matrix such that $UU^T = U^T U = \mathbf{I}$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix with the eigenvalues of M in the diagonals. Further, the columns of U are the eigenvectors of M.

(*a*) Prove that the largest eigenvalue of M is $\lambda_1 = 1$, and the corresponding (normalized) eigenvector is $u_1 = \frac{1}{\sqrt{\sum_{i=1}^{n} d_i}} d^{1/2}$ where $d^{1/2} \triangleq [\sqrt{d_1}, \ldots, \sqrt{d_n}]$.

(*b*) Prove that $D^{1/2}\big(\lambda_1 u_1 u_1^T\big) D^{-1/2} p^{(0)} = \pi$ for all probability distribution $p^{(0)}$.

(*c*) Consider a random walk starting with initial distribution $p^{(0)}$ after t time steps: $p^{(t)} = P^t p^{(0)}$. Since $P = D^{1/2} M D^{-1/2}$, we can write

$$\begin{aligned} P^t &= D^{1/2} M^t D^{-1/2} \\ &= D^{1/2} U \Lambda^t U^T D^{-1/2} \\ &= D^{1/2} \Big(\sum_{i=1}^{n} \lambda_i^t u_i u_i^T \Big) D^{-1/2} \; . \end{aligned}$$

Use the above representation to show that

$$p^{(t)} = \pi + D^{1/2} \sum_{i=2}^{n} \lambda_i^t u_i \left(u_i^T D^{-1/2} p^{(0)} \right) \; .$$

(*d*) In this problem, we prove a bound on the rate of convergence. Consider a random walk that starts at node i, i.e. $p^{(0)} = e_i \triangleq [\underbrace{0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0]$. Then after t time steps, the probability that the random walk is at node j is $p_j^{(t)}$ where $p^{(t)} = P^t p^{(0)}$. Prove that the distance to the stationary distribution is bounded by

$$|p_j^{(t)} - \pi_j| \;\; \leq \;\; \sqrt{\frac{d_j}{d_i}} |\lambda_2(P)|^t \; ,$$

where $\lambda_2(P)$ is the second largest eigenvalue of P.

[hint: you might find the following application of Cauchy-Schwarz inequality useful.]

$$
\begin{aligned}
\sum_{k=2}^{n} |e_j^T u_k||u_k^T e_i| &\leq \sqrt{\sum_{k=2}^{n} |e_j^T u_k|^2}\sqrt{\sum_{k=2}^{n} |u_k^T e_i|^2} \\
&\leq \sqrt{\sum_{k=1}^{n} |e_j^T u_k|^2}\sqrt{\sum_{k=1}^{n} |u_k^T e_i|^2} \\
&= 1
\end{aligned}
$$