IE 512 Graphs, Networks, and Algorithms

Midterm Solution

Problem 1

(a) Create a hyper graph with multi edges by copying all the edges in G. This gives a new graph with even degree at every node. Hence, there exists a Eulerian cycle that crosses every edge (twice) with total length $2|E|$.

(b) Traverse all the nodes in the graph by Depth First Search over the minimum spanning tree.

Problem 2

(a) Let X be a subset of A with $x \leq m$ nodes. Since M is a magic square of weight d, the sum of entries with row corresponding to X must be $\sum_{i \in X} \sum_{j=1}^{m} M_{i,j} = xd$. The positive entries lie in the columns corresponding to $N(X)$. Then, the sub-matrix with rows X and columns $N(X)$ have sum of all entries equal to xd. However, in each column sum is at most d. Therefore, there must be at least x columns (or nodes) in $N(X)$.

(b) Hall's matching theorem implies that there is a perfect matching in the bipartite graph G derived from a magic square M. Let P denote the permutation matrix corresponding to this perfect matching. Since the entries of M are integral, the positive entries are at least one. Then, $M - P$ has integral entries with row sum and columns sums all equal to $d-1$, where d is the weight of M. The theorem follows by induction on d.

Problem 3

(a) $\mu_1 = \frac{y^T B y}{y^T y}$ for the first eigenvector $y \in \mathbb{R}^{n-1}$ of B. Create a new vector $x \in \mathbb{R}^n$ by inserting a zero between the $i-1$-th and i-th entries of y. Then, $\mu_1 = \frac{y^T B y}{y^T y} = \frac{x^T A x}{x^T x} \leq \frac{v^T A v}{v^T v} = \lambda_1$ where v is the first eigenvector of A corresponding to λ_1.

(b) We can color a graph $G = (V, E)$ with maximum degree $d_{\max}$ as follows. Start from any node, which we call root, and color the node with any color. Move to its neighbors and color its neighbors with all different colors and also different from the color of the root. This is possible with $(d_{\max} + 1)$ colors. Now move to any node that is already colored and has a neighbor that is not already colored. Color its neighbors with all different colors and satisfying the coloring rule. We have enough colors such that this process continues until all nodes are colored.

(c) When the size of the graph is $n = 1$, then it has a single node and is obviously 1-colorable.

Suppose the claim is true for graph size $n \leq k$. For a graph with $k+1$ nodes, let $A(k+1)$ be its adjacency matrix with the largest eigenvalue $\lambda_1^{(k+1)}$. We know that the average degree is at most $\lfloor \lambda_1^{(k+1)} \rfloor$. It follows that there exists at least one node with degree at most $\lfloor \lambda_1^{(k+1)} \rfloor$. Let i be this node with degree d_i. Then create a new graph $G^{(k)}$ with k nodes by removing i from the original graph $G^{(k+1)}$. Let $A(k)$ be its adjacency matrix. Then, since $A(k)$ is what we get by removing a row and a column from $A(k+1)$, we know from (a) that $\lambda_1^{(k+1)} \geq \lambda_1^{(k)}$. Also, from the induction hypothesis, we know that $G^{(k)}$ is $\lfloor \lambda_1^{(k)} \rfloor + 1$ colorable.

Consider a $\lfloor \lambda_1^{(k)} \rfloor + 1$ coloring of $G^{(k)}$. From this, we get a $\lfloor \lambda_1^{(k+1)} \rfloor + 1$ coloring of $G^{(k+1)}$ by coloring the node i with a color different from its neighbors. This is always possible since, $d_i \leq \lambda_1^{(k+1)}$. This proves the desired claim.

Problem 3

(a) $M = D^{-1/2} A D^{-1/2}$, then $M d^{1/2} = d^{1/2}$, where $d_i^{1/2} = [d_1^{1/2}, \ldots, d_n^{1/2}]$. By Perron-Frobenius theorem, M had the top eigenvalue $\lambda_1 = 1$ and the corresponding eigenvector $u_1 = \frac{1}{\sum d_i} d^{1/2}$.

(b)

$$\begin{aligned} D^{1/2}(\lambda_1 u_1 u_1^T) D^{-1/2} p^{(0)} &= \frac{1}{\sum_i d_i} D^{1/2} d^{1/2} (d^{1/2})^T D^{-1/2} p^{(0)} \\ &= \frac{1}{\sum_i d_i} d \mathbb{1}^T p^{(0)} \\ &= \frac{1}{\sum_i d_i} d \\ &= \pi \ . \end{aligned}$$

(c)

$$\begin{aligned} p^{(t)} &= P^t p^{(0)} \\ &= D^{1/2} (\sum_{i=1}^{n} \lambda_i^t u_i u_i^T) D^{-1/2} p^{(0)} \\ &= D^{1/2} (\lambda_1^t u_1 u_1^T) D^{-1/2} p^{(0)} + D^{1/2} (\sum_{i=2}^{n} \lambda_i^t u_i u_i^T) D^{-1/2} p^{(0)} \\ &= \pi + D^{1/2} \sum_{i=2}^{n} \lambda_i^t u_i (u_i^T D^{-1/2} p^{(0)}) \ . \end{aligned}$$

(d)

$$\begin{aligned} \left| p_j^{(t)} - \pi_j \right| &= \left| e_j^T (D^{1/2} \sum_{k=2}^{n} \lambda_k^t u_k (u_k^T D^{-1/2} e_i)) \right| \\ &\leq \sqrt{\frac{d_j}{d_i}} |\lambda_2|^t \sum_{k=2}^{n} |e_j^T u_k u_k^T e_i| \\ &\leq \sqrt{\frac{d_j}{d_i}} |\lambda_2|^t \ . \end{aligned}$$