IE 512 Graphs, Networks, and Algorithms

Homework 1 Solution

Problem 1.1 Consider a graph $G = (V, E)$, where V is the set of vertices each corresponding to an agent, and we connect to agents with an edge if those two agents know the identity of each other. The leader wants the message to be passed to everyone, and messages can only be passes between two agents who know each other. So we need to find a **connected** subset of edges E'. Among all connected subset of edges E', we want one that maximizes the probability that the message is not intercepted. Taking the log of the given formula, we want to maximize $\sum_{(i,j)\in E'} \log(1 - p_{ij})$. We can find such E' by finding the minimum spanning tree on G with weights $-\log(1 - p_{ij})$.

Also, solving Maximum Spanning Tree problem with weights $(1 - p_{ij})$ also finds the optimal spanning tree for this problem.

Problem 1.2 On the given graph, consider any path P from node 1 to node 12. the maximum altitude is the weight of the edge on P that is maximum. Hence, finding a path with minimum maximum altitude is equivalent as the **minimax path problem** we studied in class. Realizing this, it is straight forward from page 1-10 of lecture slides that th solution P is the unique path along the miniumum spanning tree. For the particlar example graph, a minimax path is $P = 1 - 2 - 5 - 8 - 11 - 12$ with maximum altitude 5. There are many solutions with the same maximum altitude 5, for example $P = 1 - 2 - 5 - 8 - 9 - 12$.

Problem 1.3

(*a*) We want to prove that for a cycle C with the maximum weight edge e and for all MST T,

$$e \notin T \ .$$

We prove this using **proof by contradiction**. Suppose that there is a MST T with $e \in T$. We want to show that T cannot be a minimum spanning tree, and therefore there is a contradiction in our supposition. Then, it follows that e cannot be inlcuded in any MST.

Consider the cycle C, which includes e. Then there exists an edge e' in C which is not in T, because T is a tree and trees do not contain a cycle. Create a new tree T' by removing e from T and adding e'. From the assumption we know that $w_e > w_{e'}$. Hence, T' has smaller sum of weights compared to T. This violates the supposition that T is a MST, and this finishes the proof of the desired claim.

(*b*) We prove this by **contradiction**. Suppose there are two MSTs T_1 and T_2 for a given graph G. And let's assume G satisfies the assumption that any cut has a unique minimum weighted edge. Then, Consider an edge $e_1 = (i, j)$ which is included in T_1 but not in T_2. This naturally defines a cut (S, S^c). Consider breaking the edge $e_1 = (i, j)$, then S is the one side of the broken tree connected to node i, abd S^c is the other side connected to j. Then, there is at least one edge in T_2 (and not in e_1 crossing the cut (S, S^c). Let's call it e_2. Let e^* be the unique edge having minimum weight in the cut (S, S^c). Since e_1 and e_2 are diferent edges, at least one of e_1 or e_2 has to be different from e^*. Then we can always find a new tree T^* that has smaller weight than T_1 and T_2. This contradicts our supposition that both T_1 and T_2 are minimum spanning trees.

Here is how we construct the new tree. If $w(e_1) > w(e_2)$, then we remove e_1 from T_1 and add e^* to get a new tree T^* with smaller weight than T_1. If $w(e_2) > w(e_1)$, then we remove e_2 from T_2 and add e^* to get a new tree T^* with smaller weight than T_2. Otherwise, if $w(e_1) = w(e_1)$, then we remove e_1 from T_1 and add e^* to get a new tree T^* with smaller weight than T_1.

To disprove the converse: "if MST is unique, then every cut has unique minimum weight edge", we want an example where there is unique MST but there exists a cut where the minimum weight edge in that cut is not unique. Consider a triangle with three node $\{1, 2, 3\}$, and weights $w_{12} = 1$, $w_{23} = 1$, and $w_{31} = 2$. Then there is unique MST of $\{(1, 2), (2, 3)\}$, but there is a cut $(\{2\}, \{1, 3\})$ where there is no unique minimum weight edge.

(c) We prove this by **contradiction**. Suppose there are two MSTs T_1 and T_2 for a given graph G. And let's assume that G satisfies the assumption that any cycle has a unique maximum weighted edge. Since they are both spanning trees and they are not identical, the union of those two trees has at least one cycle. Also, there is at least one edge e_1 that is included in T_1 and not in T_2. Also, there is another edge e_2 which is included in T_2 and not in T_1. Let e^* be the edge with maximum wight in that cycle. Since e_1 and e_2 are not identical, we can always find a new tree T^* with weight smaller than T_1 and T_2. This contradicts our supposition that both T_1 and T_2 are minimum spanning trees.

Here is how we construct the new tree. If $w(e_1) > w(e_2)$, we remove e_1 from T_1 and add e_2 to get a new tree T^* with smaller weight than T_1. If $w(e_2) > w(e_1)$, then we remove e_2 from T_2 and add e_1 to get a new tree T^* with smaller weight than T_2. Otherwise, if $w(e_1) = w(e_1)$, then we remove e^* from T_1 and add e_2 to get a new tree T^* with smaller weight than T_1.

To disprove the converse: "if MST is unique, then every cycle has unique maximum weight edge", we want an example where there is unique MST but there exists a cycle where the maximum weight edge in that cycle is not unique. Consider a graph with 4 nodes $\{1, 2, 3, 4\}$, edges $\{(1, 2), (2, 3), (3, 4), (4, 1), (2, 4)\}$ and weights $w_{12} = w_{23} = w_{34} = 1$ and $w_{14} = w_{24} = 2$. Then there is unique MST of $\{(1, 2), (2, 3), (3, 4)\}$, but there is a cycle $\{(1, 2), (2, 4), (4, 1)\}$ where there is no unique maximum weight edge.

Problem 1.4 Suppose that G is not connected, then we claim that G' must be connected. If $G = (V, E)$ is not connected, then exists a partition $V = A \cup B$ such that they are disconnected in G. Then, in $\bar{G} = (V, \bar{E})$, for every node $a \in A$ and $b \in B$, $(a, b) \in \bar{E}$. It follows that any pair of nodes with one in A and the other in B are connected by one hop. Any pair of nodes both in A to B are connected by two hops.

Problem 1.5

(a) For all $S \in I$, we want to show that any subset of S is also in I. In other words we need to show the following lemma.

lemma. For all connected graphs $G' = (V, E')$, if we add an edge to the graph, $G'' = (V, E' \cup \{e\})$ the resulting graph is still connected (here E' denotes $E \setminus S$).

This is always true by definition.

(b) For all pair of subsets $X, Y \in I$, such that $|X| < |Y|$, we want to show that there exists an element $y \in Y \setminus X$ such that $X \cup \{y\} \in I$. In other words, we need to show the following lemma.

lemma. For all pairs of connected graphs $G'_1 = (V, E'_1)$ and $G'_2 = (V, E'_2)$ such that $|E'_1| > |E'_2|$, there exists an edge $(i, j) \in E'_1 \setminus E'_2$ such that when we remove the edge, $G''_1 = (V, E'_1 \setminus \{(i, j)\})$ stays connected.

We can prove this lemma as follows. Since G'_1 has more edges than G'_2, there exists a cycle $C = \{e_1, e_2, \ldots, e_k\}$ in G'_1 such that $C \subseteq E'_1$ but $C \nsubseteq E'_2$. Then, there exists an edge e_i that is in C but not in E'_2. This edge is $e_i \in E'_1 \setminus E'_2$ and when we remove it from the cycle, the graph G'_1 stays connected.