IE 512 Graphs, Networks, and Algorithms

Homework 2 Solution

Problem 2.1 We use a similar algorithm as the stable marriage problem, but modified slightly for this particular problem.

- WHILE there exists colleges engaged to strictly less than c_i students
- Each 'unengaged' students 'propose' to the most favorable college he has not proposed to yet. Among the 3rd place colleges, a student chooses one at random.
- Each college chooses the most favorable student out of those who are proposing and his current students he is engaged to, and gets engaged to the best c_i students
- RETURN the resulting engagements

We prove that the resulting 'matching' is 'stable' in the sense that no unmatched student and college pair would prefer swapping partners. Suppose student A is matched to college a but prefers college b over a. This happens only if b is A's first choice, or if b is A's second choice and a is one of third choices. Also, college b is matched to student B but prefers student A over B. Then, the pair (A, b) is unstable. From the outcome of the algorithm we know

- A has proposed to a
- B has proposed to b
- AND A has never proposed to b (otherwise A and b will be matched)

This implies that A proposed to a before b, which only happens if A preferes $a > b$. This contradicts the assumption that A prefers $b > a$. Therefore, there cannot be a unstable pair in the resulting matching of the proposed algorithm.

Problem 2.2 Start with an empty graph G_0 at time $t = 0$. At each iteration, add one edge that has the smallest t_{ij} among the ones that have not been added yet. Let G_t be the resulting graph at time t, that has t edges. Run maximum bipartite matching algorithm, and check if there exists a matching of size m. If there is a matching of size m, then this is the matching that minimizes the maximum time to reach a crime scene. If there is no matching of size m, then repeat the procedure by adding another edge with minimum weight.

We can prove that this algorithm finds the correct solution. Let's say the first time there is a matching of size m is at time t. If there is another matching that has smaller maximum time to reach a crime scene, then all the edges in the matching must be included in G_{t-1}, by construction. The fact that there is no matching in G_{t-1} of size m implies that there is no matching that has smaller maximum time to crime scene than the one found by the algorithm.

Problem 2.3

(a) Let E be the set of cells that are not holes. Let I_1 be the set of subsets of cells, such that no two cells are chosen from the same row. Le tI_2 be the set of subsets of cells, such that no two cells are chosen from the same column. We show that (E, I_1) satisfy the exchange property and the same for (E, I_2) follows similarly.

For all $X, Y \in I_1$ with $|X| < |Y|$, there exists a row such that a cell (i, j) is chosen in that row in Y but not in X. Then, it is clear that $X \cup \{(i, j)\} \in I_1$, and this proves the exchange property and that (E, I_1) is a matroid.

(b) By definition, the intersection of I_1 and I_2 is the set of all possible placements of rooks where they do not overlap in any rows or columns.

Problem 2.4

(a) Let M^* denote a perfect matching in $G = (A, B, E)$. For every $X \subseteq A$, the following is true.

$$|X| = |\{b \in B \mid \exists a \in X \text{ such that } (a, b) \in M^*\}| \leq |N(X)|$$

(b) We provide a proof by contradiction.

Suppose condition (1) holds, but there is no perfect matching. Let M be a maximum matching that is not perfect. Then there exists a node a in A that is free. Consider all alternating paths of length at least two starting from a, and let U and V be the set of nodes in (at least one of) the alternating paths in A and B respectively.

lemma. $|N(U \cup \{a\})| = |U \cup \{a\}| - 1$

This lemma proves that condition (1) is violated so it is a contradiction. We are now left to prove the lemma.

claim 1. $N(U) = V$
If there is any neighbor b of U outside V, then this creates an augmenting path, which violates the assumption that M is a maximum matching. Hence,

$$N(U) \subseteq V, \text{ and } |N(U)| \leq |V|$$

All nodes in V are matched w.r.t M, since they are a part of alternating paths of length a least two. And the nodes that are matched to V must be in U. Hence,

$$|V| \leq |U| \leq |N(U)|$$

. This proves the claim.

claim 2. $N(a) \subseteq V$
If there was any neighbor b of a outside V, then (a, b) is an augmenting path (of length one) and this violates the assumption that M is a maximum matching.

claim 3. $|N(U)| = |V|$
This follows from the proof of Claim 1.

Then, by claims 1 and 2, $N(U \cup \{a\}) = V$ and by claim 4, $|V| = |U \cup \{a\}| - 1$, which proves the lemma.

(c) We first show that for any matching M, $|M| \leq |A| - d$. Since d is the smallest integer such that the condition holds, there exists a set $Y \subseteq A$ such that $|Y| - d = |N(Y)|$. Hence, in any matching d nodes in Y cannot be matched. Therefore, $|M| \leq |A| - d$.

Now we prove that there always exists a matching of size $|M^*| = |A| - d$. construct a new graph by adding d nodes to A and connecting these spurious nodes to every node in B. Then, it is easy to see that the resulting graph satisfies Hall's matching criteria, and there is at least on perfect matching. Given this perfect matching, remove the spurious d nodes to get a matching in the original graph with size $|A| - d$.