IE 512 Graphs, Networks, and Algorithms

Homework 6 Solution

Solution 6.1

(a) Let $x_i \in \{0,1\}$ be whether an actor i is in the cast or not, and y_j be whether an investor j is funding the movie or not. Then, the total profit is $-\sum_{k=1}^{n} s_i x_i + \sum_{j=1}^{m} p_j y_j$. And we need to make sure that if investor j is funding, that all L_j actors are casted: $y_j \leq \min\{x_i\}_{i \in L_j}$. This condition can be turned into $|L_j|$ linear inequalities as follows.

$$\begin{array}{ll} \text{maximize} & -\sum_{i=1}^{n} s_i x_i + \sum_{j=1}^{m} p_j y_j \\ \text{subject to} & \text{for all } j, \;\; y_j \leq x_i, \;\; \forall i \in L_j \\ & x_i \in \{0,1\}, \;\; \forall i \\ & y_j \in \{0,1\}, \;\; \forall j \end{array}$$

(b)

$$\begin{array}{ll} \text{maximize} & -\sum_{i=1}^{n} s_i x_i + \sum_{j=1}^{m} p_j y_j \\ \text{subject to} & \text{for all } j, \;\; y_j \leq x_i, \;\; \forall i \in L_j \\ & 0 \leq x_i \leq 1, \;\; \forall i \\ & 0 \leq y_j \leq 1, \;\; \forall j \end{array}$$

(c) Let $p^* = (x^*, y^*)$ be the optimal solution to the LP relaxation and define c properly such that $c^T p$ is the objective function for $p = (x, y)$. For proof by contradiction, suppose that p^* has fractional entries, and let A be the value of the largest fractional entry of p^*. Define two vectors p_0 and p_2 as follows. p_0 is the vector you get by setting all fractional entries of p^* to zero. p_2 is the vector you get by multiplying all fractional entries of p^* by $1/A$.

By construction, p_0 is still feasible, since if there was a fractional x_i^*, then all the corresponding j's such that $i \in L_j$ should have y_j equal to zero or fractional value. Similarly, p_2 is still feasible. Let $q = p^* - p_0$. Then,

$$\begin{aligned} p_0 &= p^* - q \\ p_2 &= p^* + (\frac{1}{A} - 1)q \end{aligned}$$

Now we show the contradiction to the supposition that p^* is the optimal solution. If $c^T q$ is positive, then $c^T p_2 > c^T p^*$. Hence, p^* cannot be an optimal solution. If $c^T q$ is negative, then $c^T p_0 > c^T p^*$. Hence, p^* cannot be an optimal solution. If $c^T q$ is zero, then $c^T p_0 = c^T p^*$. Hence, there exists an optimal integral solution p_0.

(d)

$$
\begin{array}{ll}
\text{minimize} & \sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j \\
\text{subject to} & u_i - \sum_{j:i\in L_j} z_{ij} \geq -s_i \\
& v_j + \sum_{i\in L_j} z_{ij} \geq p_j \\
& 0 \leq z_{ij}, \;\; \forall j, \forall i \in L_j \\
& 0 \leq u_i, \;\; \forall i \\
& 0 \leq v_j, \;\; \forall j
\end{array}
$$

Solution 6.2 Given a primal LP

$$
\begin{array}{ll}
\text{maximize} & a_1 z_1 + \cdots + a_n z_n \\
\text{subject to} & D_{i1} z_1 + \cdots + D_{in} z_n \leq d_i, \;\; \forall i \in I \\
& D_{j1} z_1 + \cdots + D_{jn} z_n = d_j, \;\; \forall j \in E \\
& z_k \geq 0, \;\; \forall k \in N
\end{array}
$$

(a) Change the equality constraints into inequality constraints:

$$
\begin{array}{ll}
\text{maximize} & a_1 z_1 + \cdots + a_n z_n \\
\text{subject to} & D_{i1} z_1 + \cdots + D_{in} z_n \leq d_i, \;\; \forall i \in I \\
& D_{j1} z_1 + \cdots + D_{jn} z_n \leq d_j, \;\; \forall j \in E \\
& -D_{j1} z_1 - \cdots - D_{jn} z_n \leq -d_j, \;\; \forall j \in E \\
& z_k \geq 0, \;\; \forall k \in N
\end{array}
$$

Split variables without non-negativity constraints into sum of two variables:

$$
\begin{array}{ll}
\text{maximize} & \sum_{k\in N} a_k z_k + \sum_{k\notin N} a_k (z_k^+ - z_k^-) \\
\text{subject to} & \sum_{k\in N} D_{ik} z_k + \sum_{k\notin N} D_{ik} (z_k^+ - z_k^-) \leq d_i, \;\; \forall i \in I \\
& \sum_{k\in N} D_{jk} z_k + \sum_{k\notin N} D_{jk} (z_k^+ - z_k^-) \leq d_j, \;\; \forall j \in E \\
& -\sum_{k\in N} D_{jk} z_k - \sum_{k\notin N} D_{jk} (z_k^+ - z_k^-) \leq -d_j, \;\; \forall j \in E \\
& z_k \geq 0, \;\; \forall k \in N \\
& z_k^+, z_k^- \geq 0, \;\; \forall k \notin N
\end{array}
$$

(b) Write the dual LP with variables $\{w_j\}_{w\in I}, \{w_j^+\}_{j\in E}, \{w_j^-\}_{j\in E}$:

$$
\begin{array}{ll}
\text{minimize} & \sum_{k\in I} d_k w_k + \sum_{k\in E} d_k (w_k^+ - w_k^-) \\
\text{subject to} & \sum_{k\in I} D_{ki} w_k + \sum_{k\in E} D_{ki} (w_k^+ - w_k^-) \leq a_i, \;\; \forall i \in N \\
& \sum_{k\in I} D_{kj} w_k + \sum_{k\in E} D_{kj} (w_k^+ - w_k^-) \leq a_j, \;\; \forall j \notin N \\
& -\sum_{k\in I} D_{kj} w_k - \sum_{k\in E} D_{kj} (w_k^+ - w_k^-) \leq -a_j, \;\; \forall j \notin N \\
& w_k \geq 0, \;\; \forall k \in I \\
& w_k^+, w_k^- \geq 0, \;\; \forall k \in E
\end{array}
$$

(c) By setting $w_k^+ - w_k^- = w_k$, this can be further simplified as:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{k\in I} d_k w_k + \sum_{k\in E} d_k w_k = \sum_k d_k w_k \\
\text{subject to} \quad & \sum_k D_{ki} w_k \le a_i, \quad \forall i \in N \\
& \sum_k D_{kj} w_k = a_j, \quad \forall j \notin N \\
& w_k \ge 0, \quad \forall k \in I
\end{aligned}
$$

Solution 6.3

(a)

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in E} c_{ij} y_{ij} \\
\text{subject to} \quad & y_{ij} - (x_j - x_i) \ge 0 \ , \ \forall (i,j) \in E \text{ and } i \ne s, j \ne t \\
& y_{sj} - x_j \ge 1 \ , \ \forall (s,j) \in E \\
& y_{it} + x_i \ge 0 \ , \ \forall (i,t) \in E \\
& y_{ij} \ge 0 \ , \ \forall (i,j) \in E
\end{aligned}
$$

(b) Consider any path $p = (s, i_1)(i_1, i_2) \cdots (i_k, t)$ from the source to sink. The constraints for the dual problem implies that

$$\sum_{(i,j)\in p} y_{ij} = y_{si_1} + y_{i_1 i_2} + \cdots + y_{i_k t} \ge 1 + x_{i_1} + (x_{i_2} - x_{i_1}) + \cdots + (x_{i_k} - x_{i_{k-1}}) - x_{i_k} = 1 \ .$$

(c) Consider a cut (S, S^c). For all $i \in S$ set $x_i = -1$ and for all $j \in S^c$ set $x_j = 0$. Also let $y_{ij} = 1$ if $i \in S$ and $j \in S^c$, and $y_{ij} = 0$ otherwise. Then, it follows that the value of the objective function is the cut value. Also, this solution obeys the feasibility in the dual constraints.